By a *path* from A to B we mean a sequence of vertices $A, V_1, V_2, \cdots, V_n, B$ such that any two consecutive vertices in the path are connected by an edge. The *length* of the path is the number of edges in it. If we've assigned values to the edges, the *weight* of the path is the sum of the values assigned to all the edges in the path.

Lemma 1. *Given any assignment of 0,1 to the edges of an $n-$cube such that the sum of the edges of any square are even, then any two paths from A to B have the same weight modulo* 2.

Proof. Note that if we concatenate paths, the weight of the concatenation is the sum of the original weights. In this light, it suffices to prove that the weight of any loop is zero.

Now, we note that from any given vertex of an $n-$cube, there are n different vertices one can travel in, one in each principal direction (if we're representing vertices as binary strings, this each of these corresponds to flipping one bit.) We thus define n "moves" $l_1, l_2, \cdots l_n$, such that $l_k(V)$ denotes the vertex one gets after moving from V in the k-th principal direction.

A loop, then, corresponds to a starting vertex V and a sequence $a_1, a_2, \cdots, a_n$ of moves such that the number of occurrences of each l_k is even. Loops starting at a given point semigroup under concatenation of the move sequences.

Suppose now that we mod out by the equivalence relation in which two loops are equivalent if they have the same weight. Since any loop is now its own inverse, this makes our set of loops into a subgroup of the group with presentation $\langle l_1, l_2, \cdots, l_n | K \rangle$, for some set of relators K. (In particular, it is the subgroup of elements for which each of the l_i appears an even number of times.) At the very least, we note that K contains l_i^2 and $l_i l_j l_i l_j$ (the second following from the fact that the edges of any square have weights adding up to zero mod 2.) We claim that the subgroup of even degree elements of $\langle l_1, \cdots l_n | l_i^2, l_i l_j l_i l_j \rangle$ is the trivial group - this will imply that the even-degree subgroup in $\langle l_1, l_2, \cdots, l_n | K \rangle$ is also trivial, which will imply every loop has the same weight (in particular, zero.)

Because $l_i l_j l_i l_j = 1$, we can write $l_i l_j = l_j^{-1} l_i^{-1}$ and thus $l_i l_j = l_j l_i$ - in particular, our group is commutative! But this means we can rearrange any product into $l_1^{a_1} l_2^{a_2} \cdots l_n^{a_n}$. For elements of our group which correspond to loops, each of the a_i must be even, and since each l_i is its own inverse this implies all the loops correspond to the identity and thus have the same weight! (In particular, weight zero.) This proves the lemma. □

Now, we must show that there will always be a unique way to fill in the "remaining" edges of the $n-$cube. Let Q_n be the graph corresponding to our $n-$cube, and let Q_{n-1}^1 and Q_{n-1}^2 denote the two disjoint copies of Q_{n-1} which construct it. Suppose we have a good coloring of each of these, and we wish to extend to to a good coloring of Q_n.

Each edge in Q_n is either an edge in one of the copies of Q_{n-1} (which we will call a "trivial edge"), or an edge between corresponding vertices in the two copies (which we will call a nontrivial edge). We note that by the bipartite nature of Q_n, the only 4-cycle a nontrivial edge can be part of consists of two vertices in Q_{n-1}^1 and their counterparts in Q_{n-1}^2.

We can thus restate our condition that our choice of a coloring for nontrivial edges as follows:

On each edge of Q_{n-1}, we wish to place a zero or one corresponding to the mod 2 sum of the numbers on the corresponding edges in Q^1_{n-1} and Q^2_{n-1} - we wish to place 0s or 1s on the vertices such that the number on each edge corresponds to the mod 2 sum of the numbers on the vertices bounding it. (This is the same as coloring nontrivial edges, since each nontrivial edges corresponds uniquely to one vertex and four-cycles containing nontrivial edges in Q_n correspond to pairs of nontrivial edges and the sum of the corresponding edges in Q_{n-1}.)

We now do this as follows: Choose an arbitrary vertex V of Q_{n-1} and place either a 0 or a 1 on it. (Let b denote the bit we placed on it.) We claim the remainder of the $(n-1)$cube may be filled in uniquely, as follows:

For each vertex V', choose a path from V to V' - if the weight of this path is w, we place the mod 2 sum $b \oplus w$ on V'. (This does not depend on the path chosen, due to the lemma - the weights of the edges of any 4-cycle in Q_{n-1} add up to zero because they both added up to 1 initially.) The two vertices bounding each edge must add up to that edge mod 2 (this can be seen by choosing a path leading to one of the vertices, and the same path with the edge added to it. The weights of these two paths differ by (and thus add up to mod 2) by the weight of the edge, as desired.

We thus only need to show that once V and b are chosen, the coloring is unique. But this follows, because each vertex's number is uniquely determined by any vertex sharing an edge with it and the weight of that edge. Since Q_{n-1} is connected, knowing the weights of all the edges and one vertex means there is at most one possible coloring of the rest of the vertices. QED.

(This was mostly written after 1 am and so I have no idea how followable it is.)